Contemporary Mathematics
Volume **134**, 1992

q–SPECIAL FUNCTIONS AND THEIR OCCURRENCE IN QUANTUM GROUPS

T. H. KOORNWINDER

SUMMARY

In classical Lie group theory irreducible representation spaces are often decomposed into subspaces belonging to irreducible representations of a subgroup (or a chain of subgroups). Matrix elements of the representation w.r.t. a basis obtained in this way can often be expressed in terms of special functions. In particular, *spherical functions* are the matrix elements being left and right invariant w.r.t. a subgroup forming a Gelfand pair with the big group. An elementary example is the group $SU(2)$ with diagonal subgroup isomorphic to $U(1)$. Then the matrix elements of the irreducible representations of $SU(2)$ w.r.t. a $U(1)$-basis can be expressed in terms of Jacobi polynomials and the spherical functions are Legendre polynomials. Nothing essential changes when we take the basis w.r.t. a subgroup conjugate to $U(1)$, for instance $SO(2)$.

For quantum groups the situation is different. Only few quantum subgroups are available, in general, and we cannot take conjugates of quantum subgroups. For instance, quantum $SU(2)$ has a quantum subgroup $U(1)$ (an ordinary group), but no other nontrivial quantum subgroups are known and no conjugates of $U(1)$ can be taken. The matrix elements of the irreducible representations of quantum $SU(2)$ have been computed (cf. [5], [4], [1]) as little q-Jacobi polynomials and the spherical matrix elements as little q-Legendre polynomials. (See Koelink & Koornwinder's q-special functions tutorial in these Proceedings.)

In order to say something about the missing conjugates of a quantum subgroup, we use the quantized universal enveloping algebra. Let $\mathcal{U} = \mathcal{U}_q(sl(2))$ be the algebra generated by A, A^{-1}, B, C with relations $AA^{-1} = 1 = A^{-1}A$, $AB = qBA$, $AC = qCA$, $[B,C] = (A^2 - A^{-2})/(q - q^{-1})$. $\mathcal{U}$ becomes a Hopf

1991 *Mathematics Subject Classification*. Primary 33D45, 33D80, 17B37.
This paper is in final form and no version of it will be submitted for publication elsewhere.

algebra with comultiplication $\Delta(A) = A \otimes A$, $\Delta(B) = A \otimes B + B \otimes A^{-1}$, $\Delta(C) = A \otimes C + C \otimes A^{-1}$. Call $X \in \mathcal{U}$ *twisted primitive* if $\Delta(X) = A \otimes X + X \otimes A^{-1}$. Then X has this property iff X is a linear combination of $A - A^{-1}$, B and C. This three-dimensional space is a kind of quantum analogue of the Lie algebra $sl(2)$.

The Hopf $*$-algebra $\mathcal{A}$ of "polynomial functions" on quantum $SU(2)$ is in doubly non-degenerate Hopf algebra duality with $\mathcal{U}$. Call $a \in \mathcal{A}$ *left (right) invariant* w.r.t. a twisted primitive element $X \in \mathcal{U}$ if $(X \otimes \mathrm{id})(\Delta(a)) = 0$ (respectively $(\mathrm{id} \otimes X)(\Delta(a)) = 0$).

Theorem ([2], [3]). *Let X_1 and X_2 be twisted primitive elements in $\mathcal{U}$. Then the elements of $\mathcal{A}$ being left invariant w.r.t. X_1 and right invariant w.r.t. X_2 form a subalgebra generated by a single element ρ of $\mathcal{A}$ which is quadratic in the generators of $\mathcal{A}$. The intersection of this subalgebra with the subspace of $\mathcal{A}$ belonging to an irreducible representation of quantum $SU(2)$ of odd dimension $2l+1$ is one-dimensional and spanned by a certain Askey-Wilson polynomial of degree l and argument ρ.*

References

1. T. H. Koornwinder, *Representations of the twisted $SU(2)$ quantum group and some q-hypergeometric orthogonal polynomials*, Nederl. Akad. Wetensch. Proc. Ser. A **92** (1989), 97–117.
2. ———, *Orthogonal polynomials in connection with quantum groups*, Orthogonal polynomials: Theory and Practice, P. Nevai (ed.), NATO ASI Series C, Vol. 294, Kluwer Academic Publishers, 1990, pp. 257–292.
3. ———, *Askey-Wilson polynomials as zonal spherical functions on the $SU(2)$ quantum group*, preprint, CWI Rep. AM-R9013, Amsterdam.
4. T. Masuda, K. Mimachi, Y. Nakagami, M. Noumi and K. Ueno, *Representations of the quantum group $SU_q(2)$ and the little q-Jacobi polynomials*, J. Funct. Anal. **99** (1991), 127–151.
5. L. L. Vaksman and Ya. S. Soĭbel'man, *Algebra of functions on the quantum group $SU(2)$*, Functional Anal. Appl. **22** (1988), 170–181.

CWI, P.O. Box 4079, 1009 AB Amsterdam, The Netherlands